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Parameter equivalence for the Gardner and van Genuchten soil hydraulic conductivity functions for steady vertical flow with inclusions

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Abstract

The analytic element method is well suited for the Gardner hydraulic conductivity function, but is limited in describing real soils. Therefore, parameter equivalence between the van Genuchten and Gardner hydraulic conductivity functions is explored for the case of steady vertical flow through a homogeneous medium with a single inclusion, i.e., a binary soil. The inclusion has different hydraulic parameters than the background medium. Equivalence is established using three methods: (1) effective capillary drive; (2) capillary length; (3) and a least-squares optimization method that aims to fit a Gardner function to a corresponding van Genuchten function by minimizing the difference in log conductivity over a specified pressure range. Comparisons between hydraulic models are made based on scatterplots of pressure head and the vertical Darcian flux obtained using a finite-element numerical solution with both constitutive relations. For applicability of an equivalent Gardner function over a broad range of pressure heads, the crossover pressure must be maintained between the two parametric functions. The crossover pressure is defined as the pressure in which the hydraulic conductivity of the inclusion is equal to the background. It can be shown that a hybrid methodology of preserving the crossover pressure exactly and using the effective capillary drive will result in hydraulic parameters that are easily obtained and provide good agreement between the conductivity functions of the GR model to the VG model.

Keywords: Unsaturated flow; Equivalency; Capillary length; Effective capillary drive; Crossover pressure

1. Introduction

Parameter equivalence between the van Genuchten [26] and the Brooks–Corey [3] soil–water retention and hydraulic conductivity functions has been explored previously [11,16,27]. Each approach has preserved some characteristic about the relations, e.g., finding parameters that minimize the difference between the soil water retention curves [12] or ensuring that the total energy, represented as either the capillary length [27] or effective capillary drive [16], is preserved. Few studies have examined the parameter equivalence between the van Genuchten and Gardner [6] hydraulic conductivity functions. Examples of these equivalency models include Warrick [27], who preserved the capillary length by defining the pressure range for the flow scenario a priori. Russo [25] examined equivalent estimated parameters among van Genuchten, Brooks–Corey, and Gardner functions during inversion of outflow volume over time. Zhu et al. [31] used a vertical flux at the soil

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surface to make comparisons between van Genuchten, Brooks–Corey, and Gardner soils for evaporative studies related to upscaling of heterogeneous soils. These past parameter equivalence studies have been applied to transient flow problems and the accuracy of the equivalence relationships has been based on the agreement between predictions of evapotranspiration and runoff rate [14], infiltration rate and cumulative infiltration [16,27], or cumulative outflow [25].

Parameter equivalence studies are rarely performed for steady flow, because transient processes are thought to be more representative of vadose zone processes. However, steady flow through unsaturated soils are helpful when analyzing new modeling techniques. Recently, the analytic element method has been used successfully to model steady unsaturated flow through systems containing circular inclusions of contrasting properties [1,29]. The method relies on the Gardner soil description of a hydraulic conductivity function. The exponential dependence on the pressure head allows for simplification of Richards' equation [23] to a modified Helmhotz equation. The Helmholtz equation can be solved exactly [20,29] with unit gradient boundary conditions near infinity.

The Gardner function includes one or two constants. The Gardner 'α' (α_G), a constant with dimensions of per length, is linked to the pore size distribution, with larger values of α_G associated with coarse-textured soils [22]. The reciprocal of α_G has been interpreted as the length of the capillary fringe or the air-entry pressure value [25]. A second parameter, h_e, with units of length, can also be used to represent a non-zero air-entry pressure [18]. The Gardner function, even with its limited parameter set, has been shown to fit hydraulic conductivity data well over a limited range of pressures [9]. The main limitation on using the Gardner function is a lack of flexibility to represent soils over the full pressure range compared to other functional forms such as those of van Genuchten [26]. In addition, the hydraulic properties describing the van Genuchten soil model is represented more completely in soil databases, such as UNSODA [11]. While the applicability over small pressure head ranges limits the general use of the Gardner function for modeling unsaturated flow, the pressure head range for unsaturated steady state flow through a homogeneous medium with inclusions is generally small. For example, Warrick and Knight [29] showed for a single inclusion that the dimensionless relative matric potential, ϕ/ϕ_0 ranges from 0.25 to 4 (their Fig. 2). For the case of a dimensionless radius, $s = \alpha_G r_{incl}/2 = 1$, where r_{incl} is the inclusion radius in dimensional form, this range in relative matric potential equates to a pressure head range of ± 0.69 (cm) from the background, regardless of pressure head applied at the boundary. As a result of the limited range of pressures that arise for this flow conditions, it may be feasible to find a Gardner parameter set that matches any van Genuchten-like soil for use with the analytical element method.

The objective of this study is to investigate parameter equivalence for the van Genuchten and Gardner soil hydraulic conductivity model for the steady two-dimensional vertical flow problem with a circular inclusion, where the flow is driven by gravity alone. The comparison is based on simulated pressure heads and vertical Darcian flux over the domain between the two soil models with a numerical finite-element code. As the simplest case, we will consider only a binary soil, where the background is represented by a more coarsely textured soil than the inclusion. Unsaturated flow scenarios with binary soils have significance with respect to capillary barriers [24], flow in fractured media [15], or soils with macropores caused by burrowing animals, plant roots, and clay shrinkage [7]. Binary media have also been examined in the saturated flow literature for determination of the effective hydraulic conductivity [21], and effects of macrodispersivity to describe transport of solutes [4,5,13].

For the parameter equivalence study, we explore different methods as presented in the literature, including preservation of the effective capillary drive and capillary length, and a simple least-squares fitting algorithm that aims to minimize the difference between the logarithm of the hydraulic conductivity of the two soil models over a specified pressure range. The first two methods have a physical significance. The least-square approach does not have a physical significance, but is commonly used to fit a soil model to experimental data. The approach of using multiple methods of comparison has also been demonstrated in previous studies (e.g. [14,27]). Lastly, we demonstrate an equivalence model for a multi-inclusion example, where the range of modeled pressures are slightly larger than that for a single inclusion.

2. Theory

Constitutive relations are used to model soil hydraulic behavior through unsaturated soils. These relations relate the water content, pressure head, and hydraulic conductivity to each other and require hydraulic parameters to distinguish between the different types of soils. A brief review of the hydraulic models, their parameters, and relations to convert parameters of one hydraulic model to another are given below. Interested readers are referred to [28] for more comprehensive review of the topic.

2.1. Soil hydraulic models

The functional forms for soil hydraulic models are commonly algebraic expressions [27] because of their convenience in numerical and analytical analyses. The most common of these forms are the van Genuchten

[26], Brooks–Corey [3], and Gardner [6], although many more have been presented in the literature over the past two decades e.g., [2,10]. For our comparisons, we are interested mainly in the van Genuchten and Gardner functions. However, the analysis could be extended to include these other forms.

2.1.1. van Genuchten model

The van Genuchten (VG) [26] soil–water retention and hydraulic conductivity functions with the Mualem [17] substitution are:

$$\frac{\theta - \theta_r}{\theta_s - \theta_r} = [1 + (\alpha_{VG} h)^n]^{-m}, \tag{1}$$

$$K = K_s \frac{[1 - (\alpha_{VG} h)^{n-1}[1 + (\alpha_{VG} h)^n]^{-m}]^2}{[1 + (\alpha_{VG} h)^n]^{-m/2}}, \tag{2}$$

$$m = 1 - \frac{1}{n}, \tag{3}$$

where θ [–] is the volumetric water content at a pressure head of h [L], θ_r is the residual water content, θ_s is the saturated water content, n [–] and m [–] are empirical parameters relating to the pore size distribution, K_s [L T^{-1}] is the saturated hydraulic conductivity, K [L T^{-1}] is the unsaturated hydraulic conductivity as a function of pressure head, and α_{VG} [L^{-1}] is a constant.

2.1.2. Gardner model

The Gardner [6] hydraulic conductivity function, modified by Philip [18] to accommodate a non-zero air-entry pressure (h_e) is:

$$K = K_s \exp[\alpha_G(h - h_e)], \tag{4}$$

where α_G [L^{-1}] is a constant. Bakker and Nieber [1] used this form in formulating the analytic element method for soils of differing α_G. Russo [25] formulated the following relationship to define the water content as a function of pressure head:

$$\frac{\theta - \theta_r}{\theta_s - \theta_r} = \left[e^{\alpha_G h/2}\left(1 + \frac{\alpha_G h}{2}\right)\right]^{\frac{2}{m_G+2}}, \tag{5}$$

where m_G is a constant that takes values greater than 0 [29]. The soil hydraulic functions represented in Eqs. (4) and (5) are referred to as the Gardner–Russo (GR) relations.

2.1.3. Crossover pressure

A peculiarity of most binary soils in unsaturated flow is the crossover pressure head, i.e., the pressure head at which the hydraulic conductivity of both soil components are equal (Fig. 4). The crossover pressure head dictates that for flow near saturated conditions, water will be transported more quickly through the coarser textured soils [7]. However, at a pressure head that is drier than the crossover pressure, the finer textured soil becomes the favored conduit for fluid migration. The crossover pressure can exist for both the van Genuchten and Gardner binary soil models. The concept of crossover pressure head also applies to polyadic soils, where many crossover pressure head points could be realized in a single flow scenario (e.g., Fig. 6b of Yeh [30]). It should be noted that the crossover pressure does not exist for binary soils with equal αs. In this case, the pressure heads from both soils are scaled exactly the same, and the unsaturated hydraulic conductivity for the two soils decreases at the same rate.

2.2. Parameter equivalence

Three methods are explored to determine parameter equivalence for the hydraulic conductivity models between the VG and GR soil: (1) using the effective capillary drive defined by Morel-Seytoux et al. [16]; (2) using the capillary length defined by Warrick [27]; and (3) finding the minimum between the hydraulic conductivity functions in a least-squares optimization (LSO) procedure, similar to the method used by Lenhard et al. [12] to find Brooks–Corey parameters from a VG soil using the soil–water retention curve.

2.2.1. Effective capillary drive

Morel-Seytoux et al. [16] defined the effective capillary drive (H_{cM} [L]) as:

$$H_{cM} = \int_{-\infty}^{0} k_r(h)\,dh, \tag{6}$$

where $k_r(h)$ [–] is the relative hydraulic conductivity. For the VG soil, k_r is represented by the quotient in Eq. (2). Because the VG soil is difficult to integrate analytically, Morel-Seytoux et al. [16] performed a numerical integration. For convenience of application, the H_{cM} for the VG soil can be approximated by

$$H_{cM} = \frac{1}{\alpha_{VG}} \frac{0.046m + 2.07m^2 + 19.5m^3}{1 + 4.7m + 16m^2}, \tag{7}$$

with minimal errors (<2%) over the range of m from 0.05 to 0.7.

The H_{cM} for a Gardner soil is:

$$H_{cM} = \int_{-\infty}^{0} k_r(h)\,dh = h_e + \frac{1}{\alpha_G}. \tag{8}$$

In the first approach, we retain the same effective capillary drive for the VG and GR soils. For example, consider the VG parameters of a Berino fine sand (Table 1). Using Eq. (7), the effective capillary drive is 16.64 cm. Eq. (8) is then used to find the α_G that gives the same value of H_{cM} for a particular h_e. To estimate the h_e, we will define it as the pressure head when the relative hydraulic conductivity is 0.9. The estimation procedure requires a polynomial to be fit to values of dimensionless pressure head (i.e., $\alpha_{VG} * h$ [27]) over a range of m:

Table 1
Soil parameters and hydraulic functions used (after Hills et al. [8])

	α (cm^{-1})	m	θ_s	θ_r	K_s (cm s^{-1})
Glendale clay loam	0.0104	0.283	0.469	0.106	1.52(10)$^{-4}$
Berino loamy fine sand	0.028	0.553	0.366	0.0286	6.26(10)$^{-3}$

$$h_e \cong \frac{1}{\alpha_{VG}}(-2.0692m^3 - 4.4099m^2 - 1.5366m - 0.1504)^2. \tag{9}$$

The range of m considered was from 0.2 to 0.8 (n = 1.2–5), which represents a range for typical soils [16]. Appendix A demonstrates how the polynomial was obtained and gives results for other values of relative hydraulic conductivity. For our comparison study, the values for m and α_{VG} in Table 1 were used to obtain the air-entry pressure for the Berino fine sand, which is -3.36 cm. Using Eq. (8), the equivalent α_G is 0.0753 (cm^{-1}), which leads to good agreement between the two soil models over the pressure range of 0 to -70 cm (Fig. 1). Table 2 lists the results of parameter equivalence for both the Berino and Glendale soils.

2.2.2. Capillary length

Philip [19] and Warrick [27] defined a capillary length, λ_c [L] as:

$$\lambda_c = \frac{1}{k_{wet} - k_{dry}} \int_{h_{dry}}^{h_{wet}} k(h)\,\mathrm{d}h, \tag{10}$$

where k_{wet} and k_{dry} are the relative hydraulic conductivity values evaluated at h_{wet} and h_{dry}, respectively. The values of h_{wet} and h_{dry} are problem specific and define the range of pressures encountered in a flow scenario. For the GR soil, the capillary length is:

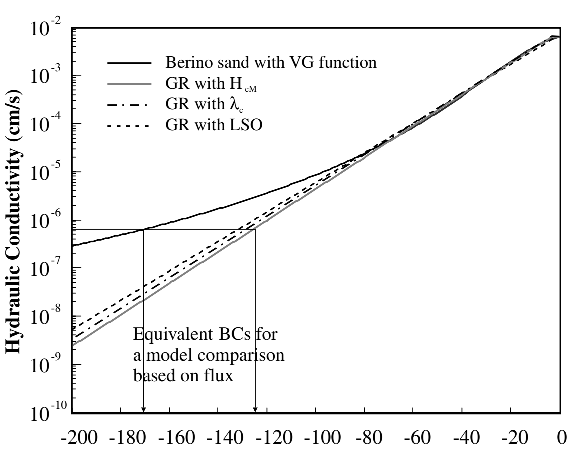

Fig. 1. Hydraulic conductivity functions for the Berino fine sand, represented as a VG soil, a GR soil using the effective capillary drive method, a GR soil using capillary length, and a GR soil using the least-squares optimization method.

$$\lambda_c = \frac{1}{k_{wet} - k_{dry}} \int_{h_{dry}}^{h_{wet}} k(h)\,\mathrm{d}h = \frac{1}{\alpha_G}, \quad h_e < h_{wet} < -\infty$$
$$= (h_e - h_{wet}) + \frac{1}{\alpha_G}, \quad 0 < h_{wet} < h_e. \tag{11}$$

To calculate the capillary length for the VG soil, Warrick [27] suggests evaluating Eq. (10) numerically. Note that the capillary drive is a special case of the capillary length for which h_{dry} approaches negative infinity and h_{wet} is 0. Table 2 lists the results of using the capillary length to obtain the α_G, and Fig. 1 shows the resulting equivalent Berino soil graphically compared to the van Genuchten curve. For the equivalency model, the pressure head values for h_{wet} to h_{dry} ranged from -65 to -63 cm. As will later be shown (e.g., Fig. 2), the pressure range for a flow scenario with a circular inclusion of r_{incl} = 1 (cm) can be quite small, but depends specifically on the soil properties. Slightly large pressure ranges can also be obtained by using larger inclusions. The simulations produced in Fig. 5a of [1] show a pressure range of approximately 9 cm.

2.2.3. Least-squares optimization (LSO)

An optimization can be performed that finds the GR parameters α_G and h_e that minimize the differences between the GR and VG hydraulic conductivity functions. As in the case with the capillary length, the pressure range over which the functions are evaluated will determine the final values for the optimized parameters. The optimization procedure aims to find the minimum of the error, E:

$$E(\boldsymbol{\beta}) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}[\ln k_{VG_i} - \ln k_{GR_i}(\boldsymbol{\beta})]^2}, \tag{12}$$

where N is the number of points over which the two functions are evaluated, k_{VG} is the relative hydraulic conductivity for the VG soil evaluated at fixed parameters of α_{VG}, and m, and k_{GR} is the relative hydraulic conductivity function for the GR soil evaluated at parameter set $\boldsymbol{\beta} = (\alpha_G, h_e)$. The parameter set can vary over parameter space and the function E will be a surface in two-dimensions. The natural logarithm of conductivity was chosen to emphasize fitting at low pressure heads [12].

As an example, consider finding equivalent GR parameters for the Berino fine sand (Table 1) over the relative conductivity range of 0.0014 to 1 using

Table 2
GR Parameters obtained for the Berino sand through different methods

Soil	k_r	H_{cM} (cm) or λ_c (cm)	h_e (cm)	α_G (cm^{-1})	Pressure range (cm)	Residual[a] (–)
Effective capillary drive						
Berino	0.9	16.64	−3.36	0.0753	–	1.1475 (from 0 to −100 cm)
Berino	0.99	16.64	0	0.0601	–	2.1351 (from 0 to −170 cm)
Glendale	0.9	16.53	−1.16	0.0651	–	2.8375 (from 0 to −100 cm)
Glendale	0.99	16.53	0	0.0605	–	8.5887 (from 0 to −170 cm)
Capillary length						
Berino	0.85	15.21	−4.26	0.0844	−65 to −63	1.5633 (from 0 to −100 cm)
Berino	0.99	34.19	0	0.0292	−171 to −169	120 (from 0 to −170 cm)
Glendale	0.99	40.44	0	0.0247	−65 to −63	3.0391 (from 0 to −100 cm)
Glendale	0.99	72.58	0	0.0138	−171 to −169	7.0469 (from 0 to −17 cm)
Least-squares optimization						
Berino	–	15.514[b]	−1.35	0.0706	0 to −100	1.1343
Berino	–	16.443[b]	0	0.0607	0 to −170	2.1057
Glendale	–	23.51[b]	0	0.0425	0 to −100	1.4898
Glendale	–	30.01[b]	0	0.0333	0 to −170	1.7245

[a] Residual is defined as: $r = \exp(\frac{1}{N}\sum|\ln k_{VG} - \ln k_{GR}|)$.
[b] H_{cM} calculated after all parameters have been determined.

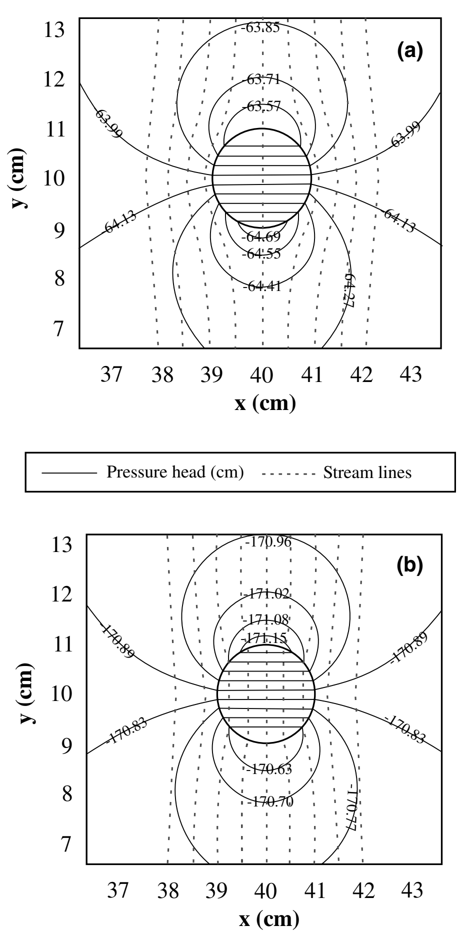

Fig. 2. Head contours from a VG soil in a FEMLAB simulation with single inclusion (a) boundary condition of $h_{inf} = -64.1$ (cm) ($k_r = 0.01$), (b) $h_{inf} = -171$ (cm) ($k_r = 0.0001$).

$N = 100$ evaluation points. The values obtained were $h_e = -1.3513$ cm and $\alpha_G = 0.07$ cm^{-1} (Table 2), and were found to vary by less than 2% when the initial guess for $\boldsymbol{\beta}$ was changed along the corners of parameter space from $\boldsymbol{\beta} = [0.001, 0]$ to $\boldsymbol{\beta} = [0.5, -20]$.

3. Methods

3.1. Flow modeling

The flow scenario for the equivalency comparison involved solving the steady state Richards' equation for the 2-D unsaturated vertical flow problem with equal constant head boundary conditions on top and bottom and no flow boundary conditions on the right and left sides:

$$0 = \frac{\partial}{\partial x}\left(K_s k_r(h)\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial z}\left(K_s k_r(h)\left(\frac{\partial h}{\partial z} + 1\right)\right), \quad (13)$$

subject to:

$$h(x, z = \text{top}) = h_{inf}, \quad (14)$$

$$h(x, z = \text{bottom}) = h_{inf}, \quad (15)$$

$$q_x(x = \text{left}, z) = 0, \quad (16)$$

$$q_x(x = \text{right}, z) = 0, \quad (17)$$

where K_s is homogeneous and isotropic within the background and within the inclusion, but the inclusion K_s is different from that of the background. The boundary pressure is h_{inf} and q_x is the flux in the x direction.

The flow system considered is a uniform background soil (Berino fine sand) with an embedded circular inclusion of finer texture (Glendale clay loam). The geometry is similar to that used in previous investigations of saturated water flow using the analytic element method [1,29]. In our model, it is necessary to place the boundaries sufficiently far as to not affect the flow field near the inclusion. The side boundaries are located at $x = 0$ and $x = 80$ (cm) and the top and bottom boundaries are at $y = -80$ and $y = 40$ (cm). The inclusion was located at (40, 10) and its radius, r_{incl}, varied from 1 to 5 (cm).

A finite-element model was written using FEMLAB v3.0 (Comsol, Los Angeles, CA) to solve Eqs. (13)–(17). FEMLAB provides the flexibility to specify soil–water constitutive relationships and the ability to couple with MATLAB (Mathworks, Natick, MA) to run a series of simulations automatically for sensitivity analyses and to conduct nonlinear least-squares optimization.

3.2. Equivalent boundary conditions

Equivalent parameters can only be defined uniquely for specific flow conditions. Therefore, the comparison will require equivalent boundary conditions between the two soil models. For a comparison based strictly on comparing pressure head measurements, h_{inf} is the same in both models. However, if comparable fluxes between the models are warranted, then a new h_{inf} is needed for the GR model that preserves a flux at the boundary equivalent to the VG model. This can be achieved by choosing a flux for a given pressure head based on the VG model. Then, the pressure head on the boundary for the GR model can be calculated as:

$$h_{\text{inf}} = \frac{\ln k_{\text{r}}}{\alpha_{\text{G}}} + h_{\text{e}}, \tag{18}$$

where k_{r} is equivalent to the dimensionless VG flux. The boundary pressure heads will be similar for both models in the pressure range where the two conductivity models are similar. However, at very low pressures, where the two hydraulic conductivity models diverge, h_{inf} can be quite different. Fig. 1 demonstrates this concept graphically.

4. Results

For the comparison of equivalent models, two test cases are presented. These cases represent conditions for which the boundary pressure head is either to the left or the right of the crossover pressure head as represented by the VG binary soil pair (approximately −131 cm). The first case is shown in Fig. 2a, in which simulated pressure head values are contoured in the immediate vicinity of the inclusion. The simulation represents the response of a binary VG soil with Berino and Glendale properties, $r_{\text{incl}} = 1$ (cm), and $h_{\text{inf}} = -64.1$ (cm). The soil has a higher pressure head at the top of the inclusion than the background; there is a decreased pressure head beneath the inclusion. The minimum and maximum pressure heads within the Berino sand are −64.8 and −63.3 (cm). The minimum and maximum pressure heads within the Glendale clay are −64.7 and −63.5 (cm). Fig. 2b shows the pressure head contours for the second case, for which the same soils are modeled with a boundary condition of $h_{\text{inf}} = -171$ (cm). Under these conditions, the pressure head is slightly higher beneath the inclusion because the Glendale clay is more conductive than the Berino sand at these low pressures. Streamlines shown in Fig. 2a and b show convergence towards the inclusion under higher flux conditions (Fig. 2a) and divergence from the inclusion under lower flux conditions (Fig. 2b).

4.1. Matching using equivalent effective capillary drive

The two flow conditions described above were modeled using the GR soil hydraulic model. The GR parameters were determined using Eq. (8) and are listed in Table 2. A direct comparison of the pressure heads at all nodes is shown for the case for which h_{inf} was held constant for the VG and GR models (Fig. 3b). An error measure was defined as the relative root-mean square difference between nodal pressures predicted using the two soil models:

$$\text{RRMSE} = \frac{1}{|\bar{p}_{\text{VG}}|}\sqrt{\frac{\sum_{i=1}^{\#\text{nodes}}(p_{\text{VG}_i} - p_{\text{GR}_i})^2}{\#\text{nodes}}}, \tag{19}$$

where $\bar{p}_{\text{VG}}$ is the mean of the pressure head of the VG simulation. The use of a relative error allows direct comparison between the model simulations. The RRMSE of the pressure heads shown in Fig. 3b is 6.47×10^{-4}. In contrast, the RRMSE of the vertical fluxes predicted with the two models is 0.057 (Fig. 3c). For transport of solutes, this difference may be significant. A simulation based on preserving the flux at the boundary (k_{r}) however, reduces the RRMSE to 0.047 (Fig. 3d). The results in Fig. 3d were obtained by modeling the flow domain with $h_{\text{inf}} = -64.51$ (cm).

Table 3 lists the results from multiple simulations after varying the value for the boundary condition and the size of the inclusion. In conditions of higher saturation, the scatterplot has a positive correlation. In drier conditions, the scatterplot has a negative correlation, which is due to the location of the crossover pressure head of the GR soil pair relative to the VG soil pair. Negative correlations happen to occur for those simulations where the boundary pressure head for the VG soils is to the left of its crossover pressure, while the GR soils are to the right of their crossover pressure. Specifically, the crossover pressure head is −131 (cm) for the VG soil and −9280 (cm) for the GR soil.

4.2. Capillary length

The steady state results demonstrating the comparisons between the VG to GR soils using the capillary length method for parameter equivalence show that there

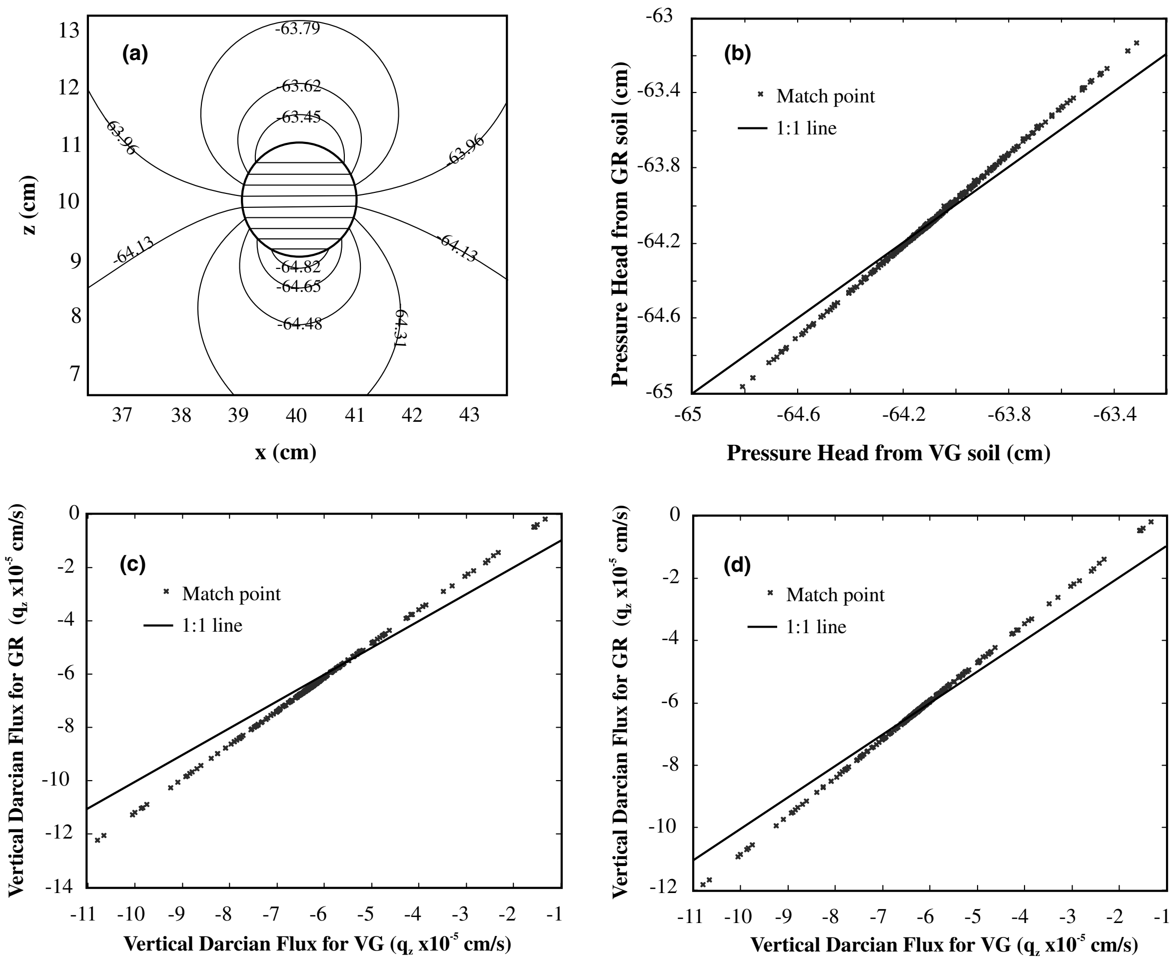

Fig. 3. (a) Head contours from running FEMLAB with GR soil preserving the effective capillary drive, with $h_{inf} = -64.211$ (cm). (b) Scatterplot of pressure head showing relations between GR and VG results, with $h_{inf} = -64.211$ (cm). (c) Scatterplot of vertical fluxes showing relations between GR and VG results, with $h_{inf} = -64.211$ (cm). (d) Scatterplot of vertical fluxes for GR and VG results with h_{inf} ($k_r = 0.01$) = −64.51 (cm).

are half as many simulations where a negative correlation exists for the head and flux data compared to the effective capillary drive method. This is due to the relative location of the GR crossover pressure head in relation to that of the VG soil pair. For the K_s values for the Berino and Glendale soil and the fitted Gardner α_G from Table 2, the Gardner crossover pressure is −241 (cm), while the VG pair is −131 (cm). When simulating the GR soil with $h_{inf} = -170$ (cm), the results are that the Berino soil is more conductive than the Glendale soil. Consequently, when running the steady state simulations with the boundary pressures derived for a flux comparison ($h_{inf} = -315$ cm), the correlations between GR and VG for the head and flux are again positive because modeling for both VG and GR occurs with a boundary condition that is to the left of the crossover pressure.

4.3. Least-squares optimization

The simulations with LSO show that all head and flux data in the scatter plot are positively correlated. Again, this is related to where the boundary pressure head is in relation to the crossover pressure. For both the dry and wet simulations, the crossover pressure is −135 (cm) for the LSO fitted GR parameters. This crossover pressure is very close to that of the VG soils, and provides a fair comparison between the two models. Additionally, the LSO simulations have a smaller RRMSE (by at least an order of magnitude) for all of the simulations based on equivalent pressure head compared to the effective capillary drive or the capillary length methods. However, for the results with a positive correlation, the effective capillary drive method produces better results for the vertical Darcian flux comparisons.

4.4. Explicit preservation of the crossover pressure

In each of the methods above, it is readily apparent that equivalency model comparisons are positively correlated for the methods that have a GR crossover pressure near the VG crossover pressure. Therefore,

Table 3
Results of FEMLAB simulations with different GR soils

BC match type	Pressure head at boundary (cm)	Flux at boundary (positive down) (cm/s)	r_{incl} (cm)	RRMSE (pressure head)	RRMSE (flux)
Effective capillary drive					
h_{inf}	−64.11	6.45×10^{-5}	1	0.00065	0.057
h_{inf}	−64.11	6.42×10^{-5}	3	0.0020	0.064
h_{inf}	−64.11	6.37×10^{-5}	5	0.0033	0.068
h_{inf}	−170.85	2.13×10^{-8}	1	0.010[a]	0.71[a]
h_{inf}	−170.85	2.16×10^{-8}	3	0.0061[a]	0.7[a]
h_{inf}	−170.85	2.13×10^{-8}	5	0.010[a]	0.71[a]
k_r	−64.51	6.25×10^{-5}	1	0.0063	0.045
k_r	−64.51	6.25×10^{-5}	3	0.0061	0.052
k_r	−64.61	6.25×10^{-5}	5	0.0062	0.055
k_r	−153.25	6.25×10^{-7}	1	0.010[a]	0.32[a]
k_r	−153.25	6.25×10^{-7}	3	0.010[a]	0.36[a]
k_r	−153.25	6.25×10^{-7}	5	0.010[a]	0.39[a]
Capillary length					
h_{inf}	−64.11	4.01×10^{-5}	1	0.0018	0.38
h_{inf}	−64.11	4.01×10^{-5}	3	0.006	0.38
h_{inf}	−64.11	3.99×10^{-5}	5	0.10	0.39
h_{inf}	−170.85	4.26×10^{-5}	1	0.0012[a]	65.74[a]
h_{inf}	−170.85	4.24×10^{-5}	3	0.0040[a]	64.83[a]
h_{inf}	−170.85	4.21×10^{-5}	5	0.0070[a]	64.22[a]
k_r	−54.56	6.25×10^{-5}	1	0.15	0.11
k_r	−54.56	6.25×10^{-5}	3	0.15	0.12
k_r	−54.56	6.25×10^{-5}	5	0.15	0.13
k_r	−315.43	6.25×10^{-7}	1	0.85	0.043
k_r	−315.43	6.25×10^{-7}	3	0.85	0.050
k_r	−315.43	6.25×10^{-7}	5	0.85	0.052
Least-squares optimization					
h_{inf}	−64.11	7.47×10^{-5}	1	0.000058	0.19
h_{inf}	−64.11	7.42×10^{-5}	3	0.00025	0.20
h_{inf}	−64.11	7.47×10^{-5}	5	0.00054	0.20
h_{inf}	−170.85	1.96×10^{-7}	1	0.00016	0.68
h_{inf}	−170.85	7.47×10^{-7}	3	0.0064	0.69
h_{inf}	−170.85	7.47×10^{-7}	5	0.0012	0.69
k_r	−66.58	6.25×10^{-5}	1	0.038	0.0021
k_r	−66.58	6.25×10^{-5}	3	0.038	0.0013
k_r	−66.58	6.25×10^{-5}	5	0.038	0.0021
k_r	−151.74	6.25×10^{-7}	1	0.11	0.031
k_r	−151.74	6.25×10^{-7}	3	0.11	0.036
k_r	−151.74	6.25×10^{-7}	5	0.11	0.038

[a] Negative correlation.

preservation of the VG crossover pressure is an important factor in the equivalency determinations. To accomplish this simply, consider forcing the GR soil pair through the VG crossover pressure (h_c) and the crossover hydraulic conductivity (K_c), as identified on a graph of VG soils (Fig. 4). The α_G and h_e can be defined using:

$$K_c = K_s \exp[\alpha_G(h_c - h_e)]. \tag{20}$$

Solving Eq. (20) for α_G and substituting the resulting expression into Eq. (8) yields:

$$H_{cM} = h_e + \frac{h_c - h_e}{\ln\left(\frac{K_c}{K_s}\right)}. \tag{21}$$

Rewriting this to define h_e gives:

$$h_e = \frac{H_{cM}\ln\left(\frac{K_c}{K_s}\right) - h_c}{\ln\left(\frac{K_c}{K_s}\right) - 1}. \tag{22}$$

Based on the VG parameters for the Berino sand and Glendale loam soils presented above, $h_c = -131$ (cm) and $K_c = 2.37\times10^{-6}$ (cm s^{-1}). Using Eq. (22), the h_e for the GR Berino soil is −0.034 (cm) and the α_G is 0.6 (cm^{-1}). Similarly, the GR Glendale soil has the properties of $h_e = -12.02$ (cm) and $\alpha_G = 0.035$ (cm^{-1}). Fig. 4 shows the resulting hydraulic conductivity functions for the GR soil pair fit to the VG pair. The calculated residuals between the VG and GR hydraulic conductivity

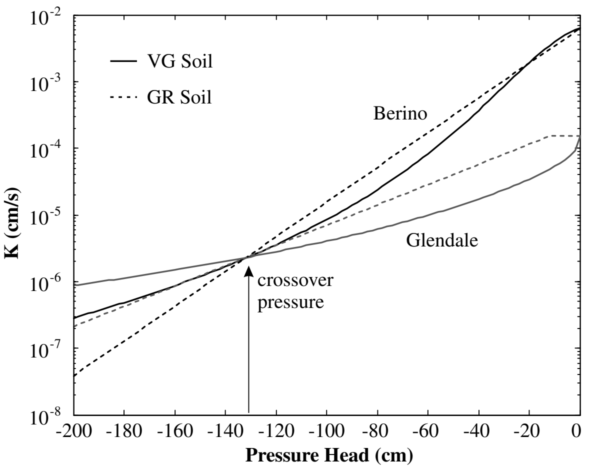

Fig. 4. A comparison of equivalence soil models using Eqs. (8) and (22).

models over the range from −170 to 0 (cm) are 1.586 and 2.109 for the Berino and Glendale soils, respectively.

4.5. Multiple inclusions

As a last example, we will consider a steady flow model with multiple circular inclusions. The purpose of this example is to demonstrate that the equivalency methods used in the single inclusion example can be extended to more complicated flow scenarios. Additionally, we will make use of Eqs. (8) and (22), such that the equivalency parameters can be obtained in the most straight-forward manner and the crossover pressure is equivalent between the two soil models. Using Eqs. (8) and (22) is simply a modified version of the effective capillary drive method presented in Section 4.1 and verified in Section 4.4. For this simulation, the domain is the same as that of the single inclusion simulations. The inclusions, with radii from 1 to 5 (cm), are dispersed randomly throughout the domain. Just as in the single inclusion example, the background soil is the Berino sand, while the inclusions are the Glendale loam. Fig. 5a shows the resulting pressure head distribution from

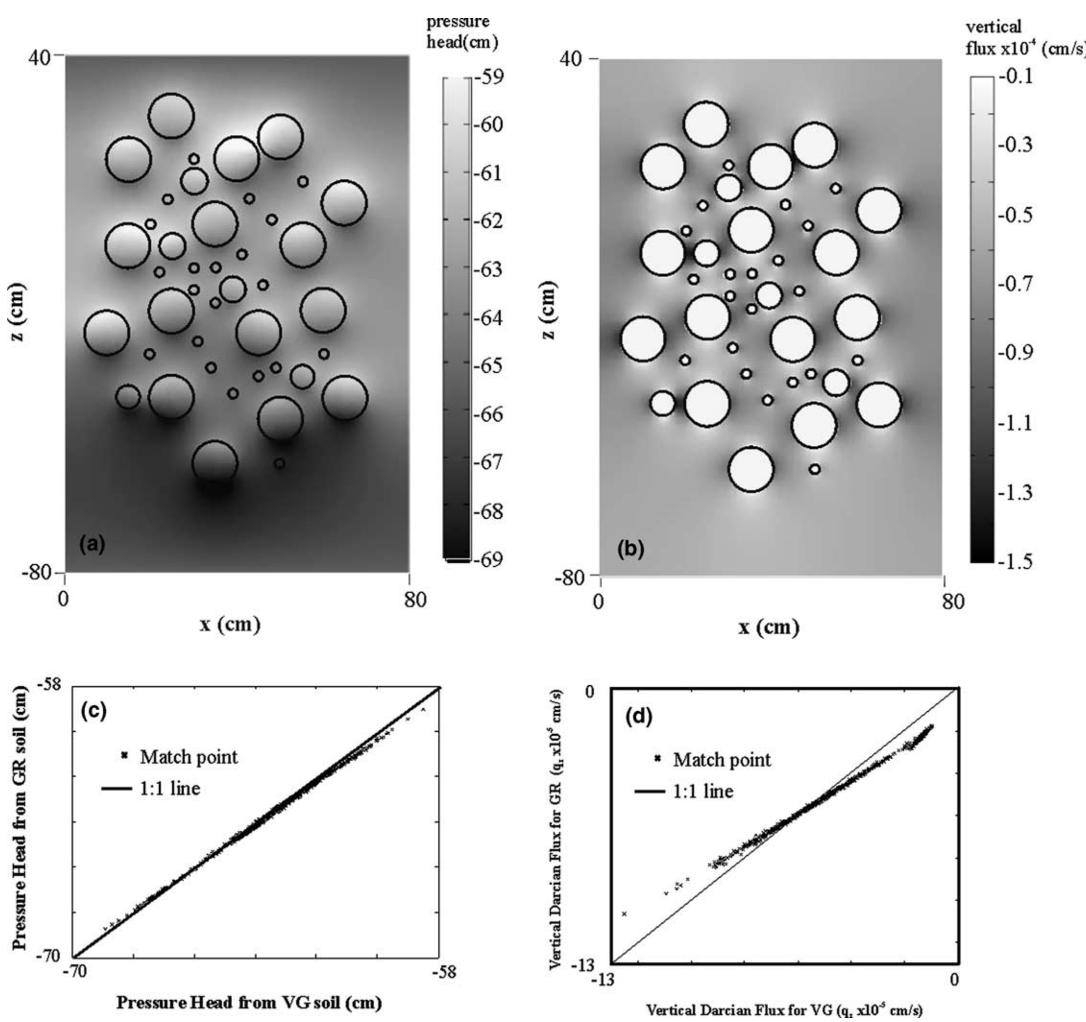

Fig. 5. (a) Pressure head contours from running FEMLAB with multiple inclusions and $h_{inf} = -64.2$ (cm). (b) Vertical flux contours from FEMLAB with $h_{inf} = -66.6$ (cm). (c) Scatterplot of pressure head for VG and GR soils with $h_{inf} = -64.2$ (cm). (d) Scatterplot of vertical fluxes for VG and GR soils with $h_{inf} = -66.6$ (cm).

running FEMLAB with the Gardner hydraulic function and $h_{\text{inf}} = -64.2$ (cm). A scatterplot of matching pressure heads (Fig. 5c) indicates that the parameter equivalence model using these formulations provides a good match. The RRMSE for the pressure heads between the VG and GR soil is 0.0033, which is in the same order as many simulations with a single inclusion. Fig. 5b shows contours of the vertical flux during a simulation where flux is preserved ($h_{\text{inf}} = -66.6$ cm). The figure shows sharp gradients that are produced between inclusions, when the inclusions are within the same horizontal plane. The corresponding scatterplot for flux comparisons (Fig. 5d), using Eq. (18) to determine the boundary condition for the Gardner simulation, also shows that the parameter equivalence works well at preserving the vertical flux at the nodes. The RRMSE for the vertical flux comparison is 0.117.

5. Conclusions

Several methods for determining a parameter equivalence between the hydraulic conductivity functions of a van Genuchten (VG) and Gardner (GR) soil, including the effective capillary drive, capillary length, and a least-squares optimization fitting, were investigated to determine which factors provided the best comparison based on scatterplots of pressure head and vertical flux. The equivalence was based on the specific problem of a binary soil with one soil type representing circular inclusions having hydraulic properties that are different from the soil representing the background. The flow scenario used in the comparison was similar to that of the inclusion problem using the analytical element method presented in Warrick and Knight [29]. For our solution, however, we used a numerical finite-element model in order to incorporate the VG soil.

The most important aspect in comparing the soil models appeared to be the preservation of the crossover pressure head. The crossover pressure head is the location where two soils of the binary pair have the same unsaturated hydraulic conductivity. In all of the examples provided, as long as the crossover pressure was maintained, the GR model performed well compared to the equivalent VG model. Ross [24] and Bakker and Nieber [1] also identified the crossover pressure head as an important point in determining flow behavior near an interface between two unsaturated soils.

Two of the methods explored, the capillary length and least-squares optimization, required a priori knowledge of the pressure head range that would be encountered in the specific flow scenario. Although it was shown that this range is effectively narrow with the center near that of the boundary pressure head, accommodating this range can be cumbersome. In contrast, the method using the effective capillary drive did not require the knowledge of the pressure head range. Additionally, the method can accommodate the crossover pressure head quite easily by finding the parameters h_e and α_G that force the GR soil model through this point. A flow simulation using the effective capillary drive equivalence model in a multiple inclusion scenario of a binary soil showed good agreement between the nodal values of the VG head and vertical flux to that of a GR model. With regards to GR representation of polyadic soils, preservation of the crossover pressure head cannot be strictly obeyed. In this case, GR parameters could be found that minimize the difference in VG and GR crossover pressure heads, while still maintaining the effective capillary drive among the soil models.

Appendix A. Effective air-entry pressure

The air-entry pressure was estimated from the van Genuchten hydraulic conductivity function by considering the pressure at which the relative conductivity fell below a predetermined value, say 0.9. However, since the conductivity function is difficult to evaluate analytically, the pressure could not be found through direct means by inversion of Eq. (2). As an alternative, the residual was evaluated through the differencing of:

$$\text{RES} = \frac{\left[1 - (h_*)^{\frac{m}{1-m}}\left[1 + (h_*)^{\frac{1}{1-m}}\right]^{-m}\right]^2}{\left[1 + (h_*)^{\frac{1}{1-m}}\right]^{-m/2}} - k_{\text{he}}, \tag{A.1}$$

where h_* is the dimensionless air-entry pressure and k_{he} is the predetermined relative hydraulic conductivity. An h_* was found by trial-and-error until the residual was less than 1e−5 for each value of m over the range from 0.2 to 0.8. For each k_{he}, a polynomial was fit to the function of m versus h_* of the form:

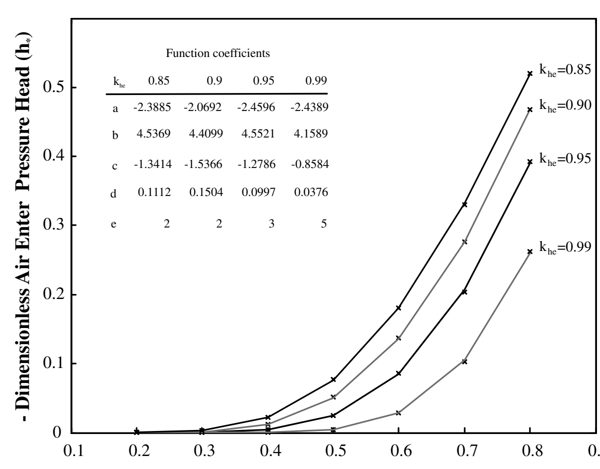

Fig. A.1. Dimensionless air-entry pressure head as a function of the van Genuchten 'm' for different values of the relative hydraulic conductivity.

$$h_* = [am^3 + bm^2 + cm + d]^e \tag{A.2}$$

and

$$h_e = \frac{1}{\alpha_{VG}} h_*, \tag{A.3}$$

where the terms a, b, c, and d were found through least-squares regression and e was determined by best judgment (Fig. A.1). The fitted polynomial was found to be in error by less than 5%, except for the lowest value of m. For $m < 0.3$, the dimensionless pressure head was less than 0.0026 for all values of k_{he} tested. As a practical limit, it is recommended that $h_e = 0$ for $m < 0.3$.

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